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Supertwistor orbifolds: gauge theory amplitudes & topological strings*

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Abstract: Witten established correspondence between multiparton amplitudes in four-dimensional maximally supersymmetric gauge theory and topological string theory on supertwistor space $\mathbb{CP}^{3|4}$. We extend Witten's correspondence to gauge theories with lower supersymmetries, product gauge groups, and fermions and scalars in complex representations. Such gauge theories arise in high-energy limit of the Standard Model of strong and electroweak interactions. We construct such theories by orbifolding prescription. Much like gauge and string theories, such prescription is applicable equally well to topological string theories on supertwistor space. We work out several examples of orbifolds of $\mathbb{CP}^{3|4}$ that are dual to $\mathcal{N} = 2, 1, 0$ quiver gauge theories. We study gauged sigma model describing topological B-model on the superorbifolds, and explore mirror pairs with particular attention to the parity symmetry. We check the orbifold construction by studying multiparton amplitudes in these theories with particular attention to those involving fermions in bifundamental representations and interactions involving U(1) subgroups.

Keywords: Brane Dynamics in Gauge Theories, Topological Strings, Conformal and W Symmetry, QCD.

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Contents

1. Introduction

In a remarkable work [1], built upon earlier observation of Nair [2], Witten discovered a twistor [3] theoretic reformulation of perturbative super Yang-Mills theory in terms of topological string theory. Specifically, Witten established a correspondence between multiparton amplitudes in $\mathcal{N}=4$ super Yang-Mills theory on $\mathbb{R}^{3,1}$ and open string amplitudes in topological B-model on Calabi-Yau supermanifold $\mathbb{CP}^{3|4}$ (Some aspects of Witten's correspondence relevant for the current work were studied in subsequent works [4]). Witten's gauge-string correspondence has successfully reproduced maximal helicity-violating (MHV) amplitudes at tree-level, and was further extended to a consistent prescription for reconstructing MHV and non-MHV amplitudes, at tree- [5] and one-loop [6] levels, out of MHV sub-amplitudes.

An immediate question is whether, in perturbation theory, it is possible to extend Witten's gauge-string correspondence to theories with supersymmetries less than $\mathcal{N}=4$. This appears not so obvious since in Witten's formulation the $\mathcal{N}=4$ supersymmetry was essential; the corresponding supertwistor space $\mathbb{CP}^{3|\mathcal{N}}$ is a Calabi-Yau (super)space only if $\mathcal{N}=4$ but not for other choices of $\mathcal{N}$. Another immediate question is to extend the pure multigluon amplitudes to those involving fermions or scalars transforming in complex representations. For those transforming in the adjoint representation, multiparton amplitudes are obtainable straightforwardly by expanding the $\mathcal{N} = 4$ Yang-Mills theory amplitudes into component fields. For multiparton amplitudes involving fermions or scalars in general representations, it is imperative to consider Yang-Mills theories with lesser or no supersymmetry. Both questions thus bring us an issue whether a map analogous to Witten's can be formulated for Yang-Mill theories with lesser or no supersymmetry, not just at tree level but also at higher orders in perturbation theory.

In this work, we dwell on this issue and make a step toward the goal. Our idea is elementary. Take an orbifold action Γ that projects covering $\mathcal{N} = 4$ Yang-Mills theory into a $\mathcal{N} < 4$ quiver gauge theory. Then, identify the corresponding operation $\widetilde{\Gamma}$ on supertwistor $\mathbb{CP}^{3|4}$ that projects symmetries and field contents in open string sector of topological B-model to those of the quiver gauge theory. As is well-known, a physical realization of the $\mathcal{N} = 4$ super Yang-Mills theory is via worldvolume theory of D3-branes placed on an ambient transverse $\mathbb{R}^6$ in type-IIB string theory. If instead we place the D3-branes at singular locus of the orbifold $\mathbb{R}^6/\Gamma$ where Γ is an element of discrete subgroup of SO(6), the worldvolume theory of D3-branes is now given in terms of gauge theories with lesser or no supersymmetry [7]. Adopting this as a route for answering the above questions, we will need to understand what the corresponding topological string theories, if they exist, are and how they are related to the topological B-model on $\mathbb{CP}^{3|4}$. We will show below that these topological string theories are defined on orbifolds of the Calabi-Yau supermanifold $\mathbb{CP}^{3|4}$. More specifically, the super-orbifolds we propose in this work are the ones in which discrete subgroup Γ of the R-symmetry group SU(4) acts on fermionic coordinates of the covering supermanifold, $\mathbb{CP}^{3|4}$. An evident but important point is that such orbifolding procedure does not violate the super Calabi-Yau condition that the covering supermanifold obeys. As said, this is a crucial ingredient for being able to define topological string theory on supermanifold.

Our result suggests that toric super-geometries and super-orbifolds are not just mathematical constructs but bears concrete physical applications in that topological B-models on such superspace are related via Witten's gauge-string correspondence to Yang-Mills theories with appropriate matter contents and supersymmetries. One would consider this relation as opening up a new avenue for supergeometries with interesting physical applications.

It should be straightforward to extend the construction to super-orientifolds — superspace obtained by orientifolding the Calabi-Yau supermanifold $\mathbb{CP}^{3|4}$. Much like defining super-orbifolds, these super-orientifolds are definable by identifying suitable projection operation as counterpart to orientifolding of the $\mathcal{N} = 4$ super Yang-Mills theory. We however postpone consideration of super-orientifolds to a separate work elsewhere.

We organize this work as follows. In section 2, we recapitulate quiver gauge theories with supersymmetry $\mathcal{N} = 2, 1, 0$ and work out the related topological string theories on appropriate super-orbifolds. We study gauged linear σ-model description of such supergeometry and explore mirrors by utilizing Landau-Ginzburg description. We also comment why it is impossible to construct $\mathcal{N} = 3$ counterparts. In section 3, to illustrate utility of such constructions, we study multiparton MHV amplitudes in quiver gauge theories, with particular attention for those involving U(1) gauge groups and fermions in complex representations. We construct them from both quiver gauge theories and from topological B-model on super-orbifolds and confirm agreement between the two results.

2. Quiver gauge theories and topological string on super-orbifolds

Consider a four-dimensional gauge theory with product gauge groups, matter fermions or scalars in complex representations and $\mathcal{N} = 4$ supersymmetries. In string theory context, a class of such gauge theory arises naturally as worldvolume theory of D3-branes on an orbifold singularity, which is known as quiver gauge theories [7]. A natural question is whether a topological string theory corresponding to quiver gauge theory does exist and, if so, what sort of operation on the supermanifold would be a counterpart to the orbifold construction. As we shall see, such operation is indeed identifiable and involves orbifolding fermionic subspace of the Calabi-Yau supermanifold $\mathbb{CP}^{3|4}$. Thus, under Witten's gauge-string correspondence, orbifolds of $\mathcal{N} = 4$ super Yang-Mills theory are mapped to topological B-model on super-orbifolds of the $\mathbb{CP}^{3|4}$.

We begin by recalling N coincident D3-branes in ambient $\mathbb{R}^{9,1}$. The R-symmetry group on the worldvolume of the D3-branes is SU(4), which is the spin cover of SO(6), the rotation group of the space transverse to the D3-brane worldvolume. The worldvolume theory of the D3-branes is four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory of gauge group $\mathrm{U}(N)$ and R-symmetry group SU(4). We shall construct quiver gauge theories as appropriate quotients of the theory by a discrete subgroup $\Gamma \in \mathrm{SU}(4)$.

2.1 $\mathcal{N} = 2$ super-orbifold

Consider the worldvolume theory on D5-branes at the orbifold $\mathbb{C}^2/\mathbb{Z}_2$, which is well known to preserve six-dimensional $\mathcal{N} = 1$ supersymmetry. The theory is of quiver type and has gauge group $\mathrm{U}(N_1)\times\mathrm{U}(N_2)$, 4 scalars transforming as $(\mathbf{N}_1, \overline{\mathbf{N}}_2)$, and 4 scalars transforming as $(\overline{\mathbf{N}}_1, \mathbf{N}_2)$. Along with gauge bosons and fermions, they fit to one $\mathcal{N} = 1$ vector multiplet of $\mathrm{U}(N_1)$, one $\mathcal{N} = 1$ vector multiplet of $\mathrm{U}(N_2)$, one $\mathcal{N} = 1$ hypermultiplet of $(\mathbf{N}_1, \overline{\mathbf{N}}_2)$, and one $\mathcal{N} = 1$ hypermultiplet of $(\overline{\mathbf{N}}_1, \mathbf{N}_2)$. Upon dimensional reduction to four dimensions, we then have $\mathcal{N} = 2$ vector multiplet of $\mathrm{U}(N_1)$ and $\mathrm{U}(N_2)$, respectively, plus $\mathcal{N} = 2$ hypermultiplet of $(\mathbf{N}_1, \overline{\mathbf{N}}_2)$ and $(\overline{\mathbf{N}}_1, \mathbf{N}_2)$, respectively. Below, we will first recapitulate construction of this quiver gauge theory in a way a direct comparison with fermionic orbifold construction is transparent. We will then construct the fermionic orbifold construction explicitly and demonstrate the equivalence.

2.1.1 Orbifolding gauge theory

Start as the covering theory from $\mathcal{N} = 4$ super Yang-Mills theory with gauge group $\mathrm{U}(2N)$, and consider the $\mathbb{Z}_2$ orbifold action on the theory defined by an element that acts simultaneously in a discrete subgroup of R-symmetry group SU(4):

$$\Gamma = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix} \in \mathrm{SU}(4) \tag{2.1}$$

and in a discrete subgroup of covering gauge group $\mathrm{U}(2N)$:

$$\gamma_\Gamma = \begin{pmatrix} +\mathbb{I}_{N_1} & \\ & -\mathbb{I}_{N_2} \end{pmatrix} \in \mathrm{U(2N)}\,. \tag{2.2}$$

Here, $N_1 + N_2 = 2N$. The large-N superconformal invariance condition requires $\mathrm{Tr}\,\gamma_\Gamma = 0$, and this is satisfied only for $N_1 = N_2 = N$. It is known that planar sector of the resulting quiver gauge theory is idential to the planar sector of the covering theory up to rescaling of gauge coupling parameters [8].

We now identify physical degrees of freedom surviving the orbifold action. Physical modes of the covering theory consist of $2(2N)^2$ gauge bosons, $6(2N)^2$ real scalar fields, and $4(2N)^2$ Weyl fermion fields, all transforming in the adjoint representation. Denote also the SU(4) indices as $A = 1, \ldots, 4$. Begin with gauge bosons. Since they are SU(4) singlets, the orbifold conditions are simply

$$\lambda = +\gamma_\Gamma \lambda \gamma_\Gamma^{-1}\,, \tag{2.3}$$

where λ represents gauge bosons collectively. This yields two sets of N^2 gauge bosons associated with two disjoint $\mathrm{U}(N)$ subgroups of $\mathrm{U}(2N)$. The resulting quiver gauge theory has gauge group $G = \mathrm{U}(N)_1 \times \mathrm{U}(N)_2$. Next, consider gauginos, which transform as **4** under SU(4). Those degrees of freedom surviving the orbifold action of eqs. (2.1), (2.2) are

$$\lambda^A = -\gamma_\Gamma \lambda^A \gamma_\Gamma^{-1} \qquad \text{for } A = 1, 2 \tag{2.4}$$

$$\lambda^A = +\gamma_\Gamma \lambda^A \gamma_\Gamma^{-1} \qquad \text{for } A = 3, 4\,, \tag{2.5}$$

where each λ^A transforms as a 2-component Weyl spinor. From eq. (2.5), we obtain 2 adjoint Weyl fermions for $\mathrm{U}(N)_1$ and 2 adjoint Weyl fermions for $\mathrm{U}(N)_2$, respectively. We also obtain 2 Weyl fermions transforming as $(\mathbf{N}, \overline{\mathbf{N}})$, and 2 Weyl fermions transforming as $(\overline{\mathbf{N}}, \mathbf{N})$. Finally, consider scalar fields. Being in antisymmetric representation[1] **6** under the SU(4) R-symmetry group, they are subject to the projection condition:

$$\lambda^{[AB]} = \sigma(A)\sigma(B)\,\gamma_\Gamma \lambda^{[AB]} \gamma_\Gamma^{-1}\,, \tag{2.6}$$

where $\sigma(i)$ equals to -1 for $A = 1, 2$ and to $+1$ for $A = 3, 4$. Explicitly,

$$\begin{aligned} \lambda^{[12]} &= +\gamma_\Gamma \lambda^{12} \gamma_\Gamma^{-1} & \lambda^{[13]} &= -\gamma_\Gamma \lambda^{13} \gamma_\Gamma^{-1} & \lambda^{[14]} &= -\gamma_\Gamma \lambda^{14} \gamma_\Gamma^{-1} \\ \lambda^{[23]} &= -\gamma_\Gamma \lambda^{[23]} \gamma_\Gamma^{-1} & \lambda^{[24]} &= -\gamma_\Gamma \lambda^{24} \gamma_\Gamma^{-1} & \lambda^{[34]} &= +\gamma_\Gamma \lambda^{[34]} \gamma_\Gamma^{-1}\,. \end{aligned} \tag{2.7}$$

Thus, we have 2 adjoint scalars for $\mathrm{U}(N)_1$, 2 adjoint scalars for $\mathrm{U}(N)_2$, 4 real scalars transforming as $(\mathbf{N}, \overline{\mathbf{N}})$, and 4 real scalars transforming as $(\overline{\mathbf{N}}, \mathbf{N})$. Putting together, such field content is precisely that of worldvolume gauge theory for D3-brane on $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$. Notice that eq. (2.7) encodes the orbifold action on $\mathbb{C}^3$ yielding $\mathbb{C}_2/\mathbb{Z}_2 \times \mathbb{C}$.

[1]Recall that **6** of SU(4) is also **6** of SO(6).

2.1.2 Orbifolding topological B-model

Next, we would like to understand how the quiver gauge theory can be mapped to an appropriate topological string theory. To match with the surviving R-symmetry group in quiver gauge theory, fermionic subspace of the supertwistor $\mathbb{CP}^{3|4}$ ought to be projected out accordingly. Since the four-dimensional fermionic subspace have SU(4) symmetry, it is quite evident that orbifolding the fermionic subspace is a viable operation of doing so. We now show that such operation produces correctly symmetries and field contents of the quiver gauge theory. We view this as an evidence that topological B-model on resulting super-orbifold is the topological string theory we are after.

The parent theory is open string sector of the topological B-model on $\mathbb{CP}^{3|4}$ [1], whose physical states are described by the (0,1)-form $\mathcal{A}$ on $\mathbb{CP}^3$, which depends on fermionic coordinates ψ_A with $A = 1, 2, 3, 4$ holomorphically. Expanding in powers of ψ's, it is given by [1]

$$\mathcal{A}(Z,\overline{Z},\psi) = \mathrm{d}\overline{Z}^{\bar{i}}\Big[A_{\bar{i}}(Z,\overline{Z}) + \psi_A\chi^A_{\bar{i}}(Z,\overline{Z}) + \frac{1}{2!}\psi_A\psi_B\phi^{[AB]}_{\bar{i}}(Z,\overline{Z}) + \\ + \frac{1}{3!}\epsilon^{ABCD}\psi_A\psi_B\psi_C\widetilde{\chi}_{\bar{i}D}(Z,\overline{Z}) + \frac{1}{4!}\epsilon^{ABCD}\psi_A\psi_B\psi_C\psi_D G_{\bar{i}}(Z,\overline{Z})\Big]\,. \quad (2.8)$$

The open string sector is described by holomorphic Chern-Simons gauge theory [9] whose action is given by

$$I = \frac{1}{2}\int_{\mathbb{CP}^{3|4}} \Omega\wedge\mathrm{Tr}\left(\mathcal{A}\overline{\partial}\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right), \quad (2.9)$$

where Ω is $(3,0|4)$ form on the Calabi-Yau supermanifold $\mathbb{CP}^{3|4}$. As explained in [1], the component (0,1)-forms $(A_{\bar{i}}, \chi^A_{\bar{i}}, \phi^{[AB]}_{\bar{i}}, \widetilde{\chi}_{\bar{i}A}, G_{\bar{i}})$ match exactly with the field contents of $\mathcal{N} = 4$ Yang-Mills supermultiplet. It was further shown that adding D-instanton effects to eq. (2.9) on $\mathbb{CP}^{3|4}$ side yields precisely the perturbative $\mathcal{N} = 4$ super Yang-Mills theory on $\mathbb{R}^4$.

Consider now the holomorphic Chern-Simons theory eq. (2.9) but now with covering gauge group U(2N). Thus, the Chan-Paton indices a, b of the component fields appearing in the ψ-expansion eq. (2.8) runs over $1, \ldots, 2N$. Consider now phase rotation of the fermionic coordinates, viz.

$$\psi_A \to e^{-i\alpha_A}\psi_A\,, \qquad Z_i, \overline{Z}_{\bar{i}} \quad \text{intact}\,. \quad (2.10)$$

We will choose the phases so that $\alpha_1 + \cdots + \alpha_4 = 0$ modulo 2π. With such a choice, the measure $\mathrm{d}^3Z\mathrm{d}^4\psi$ is invariant under the rotation eq. (2.10). It also implies that the $(3,0|4)$-form Ω is invariant under the phase rotation eq. (2.10) since it is locally given by the measure $\mathrm{d}^3Z\mathrm{d}^4\psi$.

Since $A_{\bar{i}}$ is inert under eq. (2.10), in order for $\mathcal{A}$ to transform as (0,1)-form, the component field $\chi^A_{\bar{i}}$ ought to transform as $\chi^A_{\bar{i}} \to e^{+i\alpha_A}\chi^A_{\bar{i}}$. Other component (0,1)-form fields would also transform appropriately, which we can read off straightforwardly from eq. (2.8). Combined with the aforementioned U($2N$) Chan-Paton factors, the rotation in fermionic coordinates eq. (2.10) would transform an open string field Ψ as

$$\Big|\Psi, ab\Big\rangle \to (\gamma_{g(\alpha)})_{aa'}\Big|g(\alpha)\cdot\Psi, a'b'\Big\rangle(\gamma^{-1}_{g(\alpha)})_{b'b}\,. \tag{2.11}$$

Here, $\alpha = (\alpha_1, \ldots, \alpha_4)$ and $g(\alpha)\cdot\Psi$ refers to the open string field transformed as eq. (2.10). Note that this is consistent with the fact that, as line bundles over $\mathbb{CP}^3$, fermionic coordinates are sections of type $\psi_A \sim \mathcal{O}(+1)$ and component fields are sections of type $A \sim 0, \chi \sim \mathcal{O}(-1), \phi \sim \mathcal{O}(-2), \widetilde{\chi} \sim \mathcal{O}(-3)$ and $G \sim \mathcal{O}(-4)$.

In the component expansion eq. (2.8), $A, B, \ldots$ are indices along fermionic directions in $\mathbb{CP}^{3|4}$ and hence transform as **4** under the transformation group SU(4). This SU(4) is isomorphic to the R-symmetry group in the gauge theory side. It is therefore natural to identify the orbifold action eq. (2.1) with the corresponding action in the twistor space:

$$\psi_A \to -\psi_A \quad \text{for } A = 1,2\,; \qquad \psi_B \to +\psi_B \quad \text{for } B = 3,4\,. \tag{2.12}$$

This is precisely transformations of the sort eq. (2.10). Once we identify the $\mathbb{Z}_2$ action as in eq. (2.12), it then determines the $\mathbb{Z}_2$ action on the components fields according to eq. (2.11). Recall that component fields are in different representations of SU(4), viz. $\phi_{\bar{i}}^{[AB]}$ transforming in **6**, $\widetilde{\chi}_{\bar{i}A}$ transforming in $\overline{\mathbf{4}}$, and $G_{\bar{i}}$ transforming as a singlet. After taking $\gamma_{g(\alpha)}$ the same as γ_Γ in eq. (2.2) and quotienting by the $\mathbb{Z}_2$ action according to the representation contents under SU(4), we see that the transformation rule eq. (2.11) coincides exactly with the orbifold conditions eqs. (2.3), (2.5), (2.6) in the Yang-Mills theory side. Therefore, component fields surviving the fermionic orbifold projection in the holomorphic Chern-Simons theory are precisely the component fields of the quiver gauge theory surviving the $\Gamma = \mathbb{Z}_2$ orbifold projection in $\mathcal{N} = 4$ super Yang-Mills theory, as recapitulated in the previous section.

One can consider orbifold actions by other subgroups of SU(4). The $\Gamma = \mathbb{Z}_2$ construction worked out above demonstrates it obvious that a chosen orbifold action acts the same way for the R-symmetry representations in $\mathcal{N} = 4$ super Yang-Mills theory and for the fermionic coordinates in the twistor superspace $\mathbb{CP}^{3|4}$. Equivalently, the orbifold action on the component fields of D3-brane worldvolume theory is the same as that on the component fields of $\mathcal{A}$ in the holomorphic Chern-Simons theory eq. (2.9).

2.1.3 Linear sigma model on $\mathbb{Z}_2$ super-orbifold

In the previous subsection, we prescribed topological B-model on super-orbifold $\mathbb{CP}^{3|4}/\mathbb{Z}_2$. Utilizing mirror symmetry, one can map it to topological A-model on a mirror Calabi-Yau superspace. Identification of the latter would be of interest for several reasons. First, for the parent theory, it was argued that topological B-model on $\mathbb{CP}^{3|4}$ is mirror to topological A-model on a quadric in $\mathbb{CP}^{3|3} \times \mathbb{CP}^{3|3}$. A feature of this topological A-model as opposed to the B-model is that parity symmetry[2] (under which helicities are reversed) is manifest [10].

An interesting question is whether mirror of the topological strings on super-orbifold continues to have a geometric realization as its parent theory does and, if so, whether

[2]For the consideration of parity symmetry in topological B-model, see [12] and references therein.

various discrete symmetries such as the parity are manifest. Second, though the super-orbifold projection on topological open string sector is well defined, it is not yet clear to us how to identify topological closed string sector on a super-orbifold. The reason is because of potential existence of twisted sectors. A viable route of identifying the twisted sectors is mapping the topological string to its mirror dual. If the mirror theory happens to admit a geometric realization on a regular (super)manifold, then identification of the closed string sector would be free from such ambiguity. The full spectrum on the mirror can then be mapped back to that on the original super-orbifold.

With such motivations, we now examine mirror of the super-orbifold $\mathbb{CP}^{3|4}/\mathbb{Z}_2$. We shall adopt the strategy of [10], and describe the super-orbifold in terms of σ-model. For the covering theory, $\mathbb{CP}^{3|4}$ is described by the U(1) gauged linear σ-model involving 4 bosonic and 4 fermionic chiral superfields representing the coordinates Z^I and Θ^I of $\mathbb{CP}^{3|4}$, where $I = 0, 1, 2, 3$. These fields are subject to the D-term constraint:

$$\sum_{I=0}^{3} |Z^I|^2 + \sum_{I=0}^{3} |\Theta^I|^2 = r\,,$$

where r is the real part of the Fayet-Iliopoulos parameter $r + i\theta$ and U(1) charges are assigned as $Q(\Phi^I) = Q(\Theta^I) = 1$ so that the Calabi-Yau condition $\mathrm{Str}\, Q = 0$ is obeyed. Adopting the method of [10], mirror of the topological B-model on $\mathbb{CP}^{3|4}$ can be constructed by studying the Landau-Ginzburg model whose partition function is defined by

$$\begin{aligned} Z[r] &\equiv \langle 1 \rangle \\ &= \int \prod_{I=0}^{3} \mathrm{d}Y_I \mathrm{d}Z_I \mathrm{d}\eta_I \mathrm{d}\chi_I \, \exp\left[e^{-Y_I} + e^{-Z_I}(1 + \eta_I \chi_I)\right] \delta\left(\sum_{J=0}^{3}(Y_J - Z_J) - r\right). \end{aligned} \quad (2.13)$$

Here, the variables in the gauged σ-model are changed to $Y_I = |\Phi_I|^2 + \cdots$, $Z_I = -|\Theta_I|^2 + \cdots$, and η_I, χ_I are a pair of fermionic variables conjugate to each Z_I's introduced in the course of the change of variables.

The work [10] argued that parity symmetry of the $\mathcal{N} = 4$ Super Yang-Mills theory is encoded by the $\mathbb{Z}_2$ symmetry

$$\begin{aligned} \mathtt{P}\,:\; Y_I &\leftrightarrow Z_I \\ \eta_I &\rightarrow e^{-Y_I}\chi_I \\ \chi_I &\rightarrow e^{+Z_I}\eta_I \\ r &\rightarrow -r\,, \end{aligned}$$

and demonstrated that mirror on which topological A model is defined by a quadric in $\mathbb{CP}^{3|3} \times \mathbb{CP}^{3|3}$. On the mirror side, the parity symmetry is realized as exchange symmetry of the two $\mathbb{CP}^{3|3}$'s and reversing sign of the moduli $r \leftrightarrow -r$.

Gauged linear σ-model describing the topological B-model on the super-orbifold $\mathbb{CP}^{3|4}/\mathbb{Z}_2$ is readily constructed. To this end, consider a $\mathrm{U}(1)\times\widetilde{\mathrm{U}(1)}$ gauged linear σ-model involving 4 bosonic chiral superfields $\Phi^0, \ldots, \Phi^3$ and 5 fermionic chiral superfields $\Theta^0, \ldots, \Theta^4$. We assign the $\mathrm{U}(1) \times \widetilde{\mathrm{U}(1)}$ charges as $(+1, 0)$ for $\Phi^0, \ldots, \Theta^0, \Theta^1$, as $(+1, +1)$

for Θ^2, Θ^3, and as $(0, -2)$ for Θ^4. The charge assignment is consistent with the Calabi-Yau condition $\mathrm{Str}\, Q = 0$ and $\mathrm{Str}\, \widetilde{Q} = 0$. The D-term conditions now read

$$\begin{aligned} |\Phi^0|^2 + |\Phi^1|^2 + |\Phi^2|^2 + |\Phi^3|^2 + |\Theta^0|^2 + |\Theta^1|^2 + |\Theta^2|^2 + |\Theta^3|^2 &= r \\ |\Theta^2|^2 + |\Theta^3|^2 - 2|\Theta^4|^2 &= t\,, \end{aligned} \tag{2.14}$$

where r, t are real parts of the two Fayet-Iliopoulos parameters for $\mathrm{U}(1) \times \widetilde{\mathrm{U}}(1)$. In the limit $t \ll 0$, eq. (2.14) indicates that $|\Theta^4|^2$ takes a large vacuum expectation value, and breaks the $\widetilde{\mathrm{U}(1)}$ gauge group to $\mathbb{Z}_2$. Since Θ^2, Θ^3 are minimally charged under the $\widetilde{\mathrm{U}(1)}$ while all others are neutral, it is seen that $t \ll 0$ limit yields $\mathbb{CP}^{3|4}/\mathbb{Z}_2$ orbifold, where the $\mathbb{Z}_2$ acts only on two fermionic coordinates Θ^2, Θ^3 and none on others.

The corresponding mirror can be constructed by considering the following one-point correlator within the covering Landau-Ginzburg theory defined by eq. (2.13):

$$\begin{aligned} Z[r,t] &\equiv \left\langle \int \mathrm{d}Z_4 \mathrm{d}\eta_4 \mathrm{d}\chi_4 \exp\left[e^{-Z_4}(1+\eta_4\chi_4)\right] \delta(2Z_4 - Z_2 - Z_3 - t) \right\rangle \\ &= \int \prod_{I=0}^{3} \mathrm{d}Y_I \prod_{J=0}^{4} \mathrm{d}Z_J \mathrm{d}\eta_J \mathrm{d}\chi_J \exp\left[\sum_{I=0}^{3} e^{-Y_I} + \sum_{J=0}^{4} e^{-Z_J}(1+\eta_J\chi_J)\right] \times \\ &\quad \times \delta(2Z_4 - Z_2 - Z_3 - t)\, \delta\left(\sum_{J=0}^{3}(Y_J - Z_J) - r\right). \end{aligned} \tag{2.15}$$

The two Fayet-Iliopoulos parameters describe deformation of the supertwistor orbifold, so r and t ought to originate from untwisted and twisted sectors, respectively.

Integrating out both (η_0, χ_0) and (η_4, χ_4) and treating e^{-Y_0} as the Lagrange multiplier Λ, the correlator eq. (2.15) is given by

$$Z[r,t] = \int \mathrm{d}\Lambda\, \Lambda \prod_{i=1}^{3} (\mathrm{d}u_i \mathrm{d}\chi_i \mathrm{d}v_i \mathrm{d}\eta_i) \exp F(u, v, \eta, \chi) \tag{2.16}$$

where

$$\begin{aligned} F(u,v,\eta,\chi) &= u_1 v_1 + u_2^2 v_2 + u_3^2 v_3 + u_1 + u_2^2 + u_3^2 + 1 + \eta_1\chi_1 + \eta_2\chi_2 + \eta_3\chi_3 + \\ &\quad + e^r v_1 v_2 v_3 + e^{-t/2} u_2 u_3\,. \end{aligned}$$

Be it as complicated, the result indicates no apparent geometric realization of the mirror. Moreover, the correlator is not even invariant under the parity symmetry, since upon treating e^{-X_0} as the lagrangian multiplier $\tilde{\Lambda}$ instead, the correlator is expressed as

$$Z[r,t] = \int \mathrm{d}\tilde{\Lambda}\tilde{\Lambda} \prod_{i=1}^{3} (\mathrm{d}u_i \mathrm{d}\chi_i \mathrm{d}v_i \mathrm{d}\eta_i) u_2^2 u_3^2 v_2^2 v_3^2 \exp G(u, v, \eta, \chi) \tag{2.17}$$

where

$$\begin{aligned} G(u,v,\eta,\chi) &= u_1 v_1 + u_2^2 v_2^2 + u_3^2 v_3^2 + v_1 + v_2^2 + v_3^2 + 1 + \eta_1\chi_1 + \eta_2\chi_2 + \eta_3\chi_3 + \\ &\quad + e^{-r} u_1 u_2 u_3 + e^{-t/2} u_2 u_3 v_2 v_3\,. \end{aligned}$$

Comparing the two alternative descriptions eqs. (2.16), (2.17), we see no sign of a discrete symmetry identifiable with the parity operation in the gauge theory side.

Though the mirror does not admit geometric realization, one can still draw a lot of information from the Landau-Ginzburg description. In case of conventional string theories defined on ordinary Calabi-Yau spaces, by studying chiral ring structure in the corresponding Landau-Ginzburg model, cohomological data of the original Calabi-Yau space could be understood easily. As such, it would be very interesting to study chiral rings in the above Landau-Ginzburg model and extract analogous cohomological data of the super-orbifold $\mathbb{CP}^{3|4}/\mathbb{Z}_2$. Among others, this may shed light on the field contents arising from the potential twisted closed string sectors, which by itself is an important issue in a complete definition of (topological) string theory on super-orbifolds.

2.2 $\mathcal{N} = 1$ super-orbifold

Extension of the orbifold construction to theories with lower supersymmetries is straightforward. Here, we would like to illustrate the construction for the simplest example yielding $\mathcal{N} = 1$ supersymmetric quiver gauge theories: supertwistor orbifold corresponding to the D3-branes localized at the fixed point of $\mathbb{C}^3/\mathbb{Z}_3$ orbifold.

2.2.1 Quiver gauge theory

As the orbifold action that would result in $\mathcal{N} = 1$ quiver gauge theory, we shall consider $\mathbb{Z}_3$ projection on $\mathcal{N} = 4$ super Yang-Mills theory with gauge group $\mathrm{U}(3N)$, defined by

$$\Gamma = \begin{pmatrix} \omega & & & \\ & \omega & & \\ & & \omega & \\ & & & 1 \end{pmatrix} \in \mathrm{SU}(4) \tag{2.18}$$

and

$$\gamma_\Gamma = \begin{pmatrix} \mathbb{I}_{N_1} & & \\ & \omega \mathbb{I}_{N_2} & \\ & & \omega^2 \mathbb{I}_{N_3} \end{pmatrix} \in \mathrm{U}(3N)\,,$$

where $\omega = \exp(2\pi i/3)$ and $N_1 + N_2 + N_3 = 3N$. The large-N superconformal invariance condition requires $\mathrm{Tr}\,\gamma_\Gamma = 0$. This condition is met only for $N_1 = N_2 = N_3 = N$, so we shall limit ourselves to such a choice of the quiver gauge group.

Field content of the resulting quiver gauge theory is determined by orbifold conditions, on which we now work out the details. Begin with the gauge bosons. Being SU(4) R-symmetry singlets, they are subject to orbifold conditions:

$$\lambda = +\gamma_\Gamma \lambda \gamma_\Gamma^{-1}\,, \tag{2.19}$$

where λ represents gauge bosons collectively. Explicitly, express a generic λ in color space as

$$\lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix}\,, \tag{2.20}$$

where each λ_{ij} $(i,j=1,2,3)$ constitutes $(N\times N)$ matrix. Then, using

$$\gamma_\Gamma \lambda \gamma_\Gamma^{-1} = \begin{pmatrix} \lambda_{11} & \omega^2\lambda_{12} & \omega\lambda_{13} \\ \omega\lambda_{21} & \lambda_{22} & \omega^2\lambda_{23} \\ \omega^2\lambda_{31} & \omega\lambda_{32} & \lambda_{33} \end{pmatrix},$$

we solve the orbifold condition eq. (2.19) and find that only the diagonal entries $\lambda_{11}, \lambda_{22}, \lambda_{33}$ survive as physical degrees of freedom. Therefore, three sets of N^2 gauge bosons survive and the resulting quiver gauge theory is a theory of gauge group $G = \mathrm{U}(N)_1 \times \mathrm{U}(N)_2 \times \mathrm{U}(N)_3$. Next, consider gauginos $\lambda^1, \ldots, \lambda^4$. The orbifold conditions are now given by

$$\lambda^i = \omega\,\gamma\lambda^i\gamma^{-1} \quad \text{for } A=1,2,3; \qquad \lambda^4 = \gamma\lambda^4\gamma^{-1}.$$

In the notation of eq. (2.20), the surviving components are $\lambda^A_{12}, \lambda^A_{23}, \lambda^A_{31}$ for $A=1,2,3$ and $\lambda^4_{11}, \lambda^4_{22}, \lambda^4_{33}$. Finally, for scalar fields $\lambda^{[AB]}$, the orbifold conditions are given by

$$\begin{aligned} \lambda^{[12]} &= \omega^2\gamma\lambda^{[12]}\gamma^{-1} & \lambda^{[13]} &= \omega^2\gamma\lambda^{[13]}]\gamma^{-1} & \lambda^{[23]} &= \omega^2\gamma\lambda^{[23]}\gamma^{-1} \\ \lambda^{[14]} &= \omega\,\gamma\lambda^{[14]}\gamma^{-1} & \lambda^{[24]} &= \omega\,\gamma\lambda^{[24]}\gamma^{-1} & \lambda^{[34]} &= \omega\,\gamma\lambda^{[34]}\gamma^{-1}\,. \end{aligned} \tag{2.21}$$

Putting the surviving components together, We find that field contents consist of one $\mathcal{N}=1$ vector multiplet and three $\mathcal{N}=1$ chiral multiplets transforming as $(\mathbf{N}, \overline{\mathbf{N}}, 1), (1, \mathbf{N}, \overline{\mathbf{N}})$ and $(\overline{\mathbf{N}}, 1, \mathbf{N})$ under the gauge group $\mathrm{U}(N)_1 \times \mathrm{U}(N)_2 \times \mathrm{U}(N)_3$. This is precisely the field contents of worldvolume theory of D3-branes on $\mathbb{C}^3/\mathbb{Z}_3$. As before, eq. (2.21) encodes the orbifold action on $\mathbb{C}^3$, yielding $\mathbb{C}^3/\mathbb{Z}_3$. Denoting the Z^1, Z^2, Z^3 the complex coordinates of $\mathbb{C}^3$, the $\mathbb{Z}_3$ action is given by $Z^m \to \omega Z^m$ $(m=1,2,3)$. Indeed, $\lambda^{[14]}, \lambda^{[24]}, \lambda^{[34]}$ correspond to complex scalars associated with the Z^1, Z^2, Z^3 directions, while $\lambda^{[12]}, \lambda^{[13]}, \lambda^{[23]}$ are associated with the complex conjugates (cf. $\phi^{[ij]} = \frac{1}{2}\varepsilon^{ijkl}\phi_{[kl]} = (\phi^{[ij]})^*$).

2.2.2 Orbifolding topological B-model

We would like to understand how the $\mathcal{N}=1$ quiver gauge theory can be mapped to an appropriate topological string theory. Again, we argue that a suitable super-orbifold can be defined which reproduces correctly the above spectrum of the quiver gauge theory.

Consider now the holomorphic Chern-Simons theory eq. (2.9) with gauge group $\mathrm{U}(3N)$, so the Chan-Paton indices a,b run over $1,\ldots,3N$. Following steps closely parallel to the $\mathcal{N}=2$ quiver gauge theory case, we find that an obvious candidate of the orbifold action on $\mathbb{CP}^{3|4}$ is

$$\psi_A \to \omega\psi_A \quad \text{for } A=1,2,3; \qquad \psi_4 \to \psi_4\,.$$

This assigns $\mathbb{Z}_3$ action on $\overline{\mathbf{4}}$ of $\mathrm{SU}(4)$, and accordingly determines orbifold action on all other component fields transforming in different representations of $\mathrm{SU}(4)$. One can see that such orbifold action on the $(0,1)$-form field $\mathcal{A}$ in eq. (2.8) projects the component fields precisely to the same as the $\mathcal{N}=1$ vector multiplet of quiver gauge theory with gauge group $G = \mathrm{U}(N)_1 \times \mathrm{U}(N)_2 \times \mathrm{U}(N)_3$ and three $\mathcal{N}=1$ chiral multiplets transforming as $(\mathbf{N}, \overline{\mathbf{N}}, 1), (1, \mathbf{N}, \overline{\mathbf{N}})$ and $(\overline{\mathbf{N}}, 1, \mathbf{N})$.

2.2.3 Linear sigma model on $\mathbb{Z}_3$ super-orbifold

It is also possible to construct a gauged linear σ-model description of the $\mathbb{CP}^{3|4}/\mathbb{Z}_3$ super-orbifold. The sigma model involves four bosonic superfields $\Phi^0, \ldots, \Phi^3$ and five fermionic superfields $\Theta^0, \ldots, \Theta^4$. This is the same field contents as the $\mathbb{Z}_2$ orbifold considered in the previous section. In order to describe $\mathbb{Z}_3$ super-orbifold, we shall need to assign $\mathrm{U}(1) \times \widetilde{\mathrm{U}(1)}$ charges $(Q, \widetilde{Q})$ differently so that the orbifolding now acts on three fermionic coordinates (instead of two as in the $\mathbb{Z}_2$ case). We assign them as $(+1, 0)$ for $\Phi^0, \ldots, \Phi^3$ and Θ^0, as $(+1, +1)$ for $\Theta^1, \Theta^2, \Theta^3$, and as $(0, -3)$ for Θ^4. Notice that the Calabi-Yau conditions, $\mathrm{Str}\, Q = 0$ and $\mathrm{Str}\, \widetilde{Q} = 0$, are satisfied by such assignment.

The D-term constraints are now given by

$$\begin{aligned} |\Phi^0|^2 + |\Phi^1|^2 + |\Phi^3|^2 + |\Theta^0|^2 + |\Theta^1|^2 + |\Theta^2|^2 + |\Theta^3|^2 &= r \\ |\Theta^1|^2 + |\Theta^2|^2 + |\Theta^3|^2 - 3|\Theta_4|^2 &= t\,. \end{aligned} \tag{2.22}$$

The moduli t arises from potential twisted closed string sector. At $t \to -\infty$, Θ^4 gets a nonzero vacuum expectation value, which breaks the $\widetilde{\mathrm{U}(1)}$ gauge group to $\mathbb{Z}_3$. Therefore, the vacuum moduli space is now reduced to $\mathbb{C}^{3|4}/\mathbb{Z}_3$ super-orbifold, where $\widetilde{\mathbb{Z}}_3$ acts on the fermionic coordinates as $\Theta^A \to \omega\Theta^A$ for $A = 1, 2, 3$. Again, this is just a fermionic counterpart of the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold and of its linear σ-model description.

Partition function of the Landau-Ginzburg $\mathbb{Z}_3$ orbifold is defined as one-point correlator of the parent Landau-Ginzburg theory:

$$\begin{aligned} Z[r,t] &\equiv \left\langle \int \mathrm{d}Z_4 \mathrm{d}\eta_4 \mathrm{d}\chi_4 \, \exp\left[e^{-Z_4}(1 + \eta_4\chi_4)\right] \delta\Big(3Z_4 - Z_1 - Z_2 - Z_3 - t\Big) \right\rangle \\ &= \int \prod_{I=0}^{3} \mathrm{d}Y_I \prod_{J=0}^{4} \mathrm{d}Z_J \mathrm{d}\eta_J \mathrm{d}\chi_J \, \exp\left[\sum_{I=0}^{3} e^{-Y_I} + \sum_{J=0}^{4} e^{-Z_J}(1 + \eta_J\chi_J)\right] \times \\ &\qquad \times \delta\Big(3Z_4 - Z_1 - Z_2 - Z_3 - t\Big) \delta\left(\sum_{I=0}^{3}(Y_I - Z_I) - r\right). \end{aligned} \tag{2.23}$$

As in $\mathcal{N} = 2$ super-orbifold case, the one-point correlator can be expressed in terms of two alternative choices of the Lagrange multiplier. Comparing the two, we again find no discrete symmetry identifiable with the parity symmetry in the gauge theory side.

2.3 Impossibility of $\mathcal{N} = 3$ super-orbifold

One might wonder if a variation of the above constructions leads to a $\mathcal{N} = 3$ super-orbifold. If so, there ought to be some $\mathcal{N} = 3$ gauge theory, dual to topological string on such super-orbifold. On the other hand, it is known that the former does not not exist and it actually is equivalent to $\mathcal{N} = 4$ theory. Thus, turned around this way, the nonexistence of $\mathcal{N} = 3$ (quiver) gauge theory proper may serve as a check-point for the consistency of the orbifold method we proposed in this work.

One way of understanding nonexistence of $\mathcal{N} = 3$ quiver gauge theories is to examine the number of adjoint fermions surviving a given SU(4) orbifolding. The number of the adjoint fermions are the fermions satisfying the orbifold condition $\lambda = \gamma\lambda\gamma^{-1}$. In order to have three such fermions and hence $\mathcal{N}=3$ supersymmetry, we should have at least three singlets for SU(4) elements associated with the orbifold action. The only such element is the identity of SU(4) and we are forced to go back to $\mathcal{N}=4$. In other words, requirement of $\mathcal{N}=3$ supersymmetry is equivalent to $\mathcal{N}=4$ supersymmetry.

2.4 Remarks on non-supersymmetric super-orbifold

One can also construct non-supersymmetric quiver gauge theories and corresponding topological string theories on appropriate super-orbifolds. The simplest procedure fitting to the pattern we constructed in the previous sections is to project the $\mathcal{N}=4$ gauge theory of gauge group $\mathrm{U}(4N)$ by the following $\mathbb{Z}_4$ orbifold action defined by simultaneous action on R-symmetry group

$$\Gamma = \begin{pmatrix} \omega & & & \\ & \omega & & \\ & & \omega & \\ & & & \omega \end{pmatrix} \in \mathrm{SU}(4) \tag{2.24}$$

and gauge group

$$\gamma_\Gamma = \begin{pmatrix} \mathbb{I}_{N_1} & & & \\ & \omega\mathbb{I}_{N_2} & & \\ & & \omega^2\mathbb{I}_{N_3} & \\ & & & \omega^3\mathbb{I}_{N_4} \end{pmatrix} \in \mathrm{U}(4N)\,. \tag{2.25}$$

Here, $\omega = e^{\frac{i\pi}{2}}$ and $N_1 + N_2 + N_3 + N_4 = 4N$. Again, to meet the large-N conformal invariance condition, we take $N_1 = N_2 = N_3 = N_4 = N$.

Spectrum of the resulting quiver gauge theory is identified as follows. From the orbifold conditions for gauge bosons:

$$\lambda = \gamma_\Gamma \lambda \gamma_\Gamma^{-1}\,,$$

we obtain 4 sets of N^2 gauge bosons associated with the gauge group $G = \mathrm{U}(N)^4$. The adjoint fermions are subject to orbifold conditions:

$$\lambda^A = \omega\gamma_\Gamma \lambda^A \gamma_\Gamma^{-1} \qquad \text{for } A = 1, \ldots, 4\,,$$

so we have 4 Weyl fermions transforming bilinearly as $(\mathbf{N}, \overline{\mathbf{N}}, 1, 1)$, $(1, \mathbf{N}, \overline{\mathbf{N}}, 1)$, $(1, 1, \mathbf{N}, \overline{\mathbf{N}})$, $(\overline{\mathbf{N}}, 1, 1, \mathbf{N})$ under four $\mathrm{U}(N)$ gauge groups. Finally, the adjoint scalar fields are subject to orbifold conditions:

$$\lambda^{[AB]} = -\gamma_\Gamma \lambda^{[AB]} \gamma_\Gamma^{-1}\,, \tag{2.26}$$

leading to six scalars transforming as $(\mathbf{N}, 1, \overline{\mathbf{N}}, 1)$, $(\overline{\mathbf{N}}, 1, \mathbf{N}, 1)$, $(1, \mathbf{N}, 1, \overline{\mathbf{N}})$ and $(1, \overline{\mathbf{N}}, 1, \mathbf{N})$. Since all the fermions transform in 'nearest-neighbor' bifundamental representations, they cannot be paired up to supermultiplets with gauge bosons nor with scalars, and it is evident that the theory is non-supersymmetric.

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Repeating the analysis as in previous setions, we find that the relevant topological string theory is the B-model defined on the super-orbifold $\mathbb{CP}^{3|4}/\mathbb{Z}_4$ obtainable from the supertwistor space $\mathbb{CP}^{(3|4)}$ by the orbifold action

$$\psi_A \to e^{-\frac{i\pi}{2}}\psi_A \qquad \text{for } A = 1, \dots, 4\,.$$

Again, it is possible to construct a linear gauged σ-model description of the above super-orbifold. The relevant model is the same one as $\mathbb{Z}_2$ and $\mathbb{Z}_3$ orbifolds, so it contains four bosonic superfields $\Phi^0, \dots, \Phi^3$ and five fermionic superfields $\Theta^0, \dots, \Theta^4$. To obtain $\mathbb{Z}_4$ super-orbifold, we would need to assign $\mathrm{U}(1) \times \widetilde{\mathrm{U}(1)}$ charges so that it now acts on all of the four fermionic directions. An obvious assignment is as $(+1, 0)$ for $\Phi^0, \dots, \Phi^3$, as $(+1, +1)$ for $\Theta^0, \dots \Theta^3$, and as $(0, -4)$ for Θ^4, respectively. Notice that $\mathrm{Str}\, Q$ and $\mathrm{Str}\, \widetilde{Q}$ all vanish, so the resulting super-orbifold is still a Calabi-Yau manifold. For bosonic $\mathbb{C}^4/\mathbb{Z}_4$ orbifold, it is known to be a well-defined Calabi-Yau fourfold, where the orbifold singularity can be deformed smoothly while preserving the Calabi-Yau conditions. This is quite analogous in situation to the $\mathbb{Z}_2$ and $\mathbb{Z}_3$ orbifolds, so we anticipate that topological B-model on $\mathbb{C}^{3|4}/\mathbb{Z}_4$ super-orbifold would provide twistor description of the above non-supersymmetric quiver gauge theory.

Another possible choice of the projection leading to a non-supersymmetric quiver gauge theory is via the action $\omega = -1$ in eqs. (2.24), (2.25), viz. $\mathbb{Z}_2$ orbifold along all four fermionic directions. The resulting quiver gauge theory have gauge group U(N)×U(N), containing a pair of fermions in bifundamental representations and scalars in adjoint representations. Again, they cannot be organized into supermultiplets, so the quiver gauge theory is non-supersymmetric. This model was considered previously in [11]. Such quiver gauge theory may opt to define the corresponding super-orbifold, viz. $\mathbb{CP}^{3|4}/\mathbb{Z}_2$. This model may, however, be potentially problematic, since we think there would be no gauged linear σ-model description obeying Calabi-Yau conditions, viz. $\mathrm{Str}\, Q$'s all vanish. This follows by inferring from known results of the bosonic counterpart. For bosonic $\mathbb{C}^4/\mathbb{Z}_2$ orbifold, it is known that although a suitable linear sigma-model can be constructed, the Calabi-Yau conditions, $TrQ = 0$, are not satisfied because of nonanalytic behavior of the topological string with respect to Kähler deformations [13]. This means that the singular Calabi-Yau manifold cannot be deformed to a smooth one. It would be interesting to demonstrate such rigidity directly for the super-orbifold $\mathbb{CP}^{3|4}/\mathbb{Z}_2$.

3. Multi-parton amplitudes in quiver gauge theories

Having constructed examples of Witten's gauge-string correspondence for theories with $\mathcal{N} < 4$ supersymmetries, in this section, we would like to test them by comparing multiparton amplitudes computed from both gauge theory and topological B-model. In this section, we shall do so for the simplest set of them. First, we shall consider MHV multiparton amplitudes involving two fermions. Since the fermions are in bifundamental representations, we expect that such amplitudes would roughly be a product of MHV amplitudes for each product gauge theories. We shall confirm that this expectation is in fact correct. Second, we shall study MHV multiparton amplitudes involving U(1) gauge groups. The quiver

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gauge theory contains overall U(1) subgroup and relative U(1)'s. As checkpoints for our orbifold constructions, we will confirm by explicit computations that overall U(1) decouples while relative U(1) gauge group yields nontrivial multiparton amplitudes exhibiting incoherence.

It is also of phenomenological interest to study multiparton amplitudes in a theory with product gauge group $G_1 \times G_2 \times \cdots$ as well as U(1)'s and fermion or scalar fields in bifundamental representations therein. The Standard Model of the strong and electroweak interactions is certainly of such structure: the gauge group is SU(3) $\times$ SU(2) $\times$ U(1), and quarks, leptons and Higgs transform in the fundamental or the bifundamental representations. Though the electroweak gauge group is spontaneously broken, at a high-energy regime well above the Fermi scale, it is natural to expect that multi-parton amplitudes are described by the theory in the phase where the gauge symmetries are unbroken. At such regime, all particles can be treated as massless and conformal invariance would play a role in governing their multi-particle amplitudes. See for example [14] for earlier studies. The $\mathcal{N} = 4$ super Yang-Mills theory contains particles of helicity $0, \pm 1/2$ and ± 1, all belonging to a single $\mathcal{N} = 4$ supermultiplet. Therefore, though the corresponding multi-parton amplitudes involve fermions as well as scalars, these particles all transform in the adjoint representation of the gauge group G. In order to study particles transforming in other representations, one needs to relax the supersymmetry from the maximal $\mathcal{N} = 4$ to lower ones. The quiver gauge theory we considered in the previous section is close in group structure and field contents to the Standard Model. In particular, matter fields transforming in bifundamental representations are readily obtainable. This suggests that quiver gauge theories may serve as a laboratory for studying features of multi-particle amplitudes in the Standard Model.

With such motivations, in this section, we shall study multi-parton amplitudes of the quiver gauge theories by computing them in the topological B-model on a chosen superorbifold. We shall compare them with known results in gauge theories with product gauge groups, including abelian groups, and with fermions. We shall first recapitulate results regarding such variants, and establish certain extensions relevant for foregoing discussions.

3.1 Parton amplitudes for helicity 0 and 1/2 adjoint representations

The MHV parton amplitude for $\mathcal{N} = 4$ super Yang-Mills theory with gauge group U(N) is given by [15, 2, 16, 1]

$$\hat{A}_n = i g_{\rm YM}^{n-2} (2\pi)^4 \delta^{(4)}(P) \delta^{(8)}(\Theta) \prod_{i=1}^{n} \frac{1}{\langle \lambda_i, \lambda_{i+1} \rangle}. \tag{3.1}$$

Here, the bosonic and fermionic momenta are

$$p^i_{a\dot{\alpha}} = \lambda_{ia} \widetilde{\lambda}_{i\dot{\alpha}}\,; \qquad \pi^i_{aA} = \lambda_{ia} \eta_{iA} \tag{3.2}$$

and $\Theta_{bA} = \sum_i \lambda_{ib} \eta_{iA}$. We can extract parton amplitudes for each helicity of the $\mathcal{N} = 4$ vector multiplet by expanding the Dirac delta function for the total fermionic momentum.

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For example, we obtain in this way the MHV amplitudes involving two external fermions and $n-2$ gluons as

$$\hat{A}_n = i g_{\rm YM}^{n-2} (2\pi)^4 \delta^{(4)}(P) \langle q\alpha\rangle^3 \langle q'\alpha\rangle \prod_{i=1}^{n} \frac{1}{\langle \lambda_i, \lambda_{i+1}\rangle} \,. \tag{3.3}$$

Here, the fermion denoted by q has the helicity $-1/2$ and the fermion q' has the helicity $1/2$ while the gluon denoted by α has the helicity -1. Similarly, we obtain MHV amplitudes involving 4 scalars of helicity 0 as

$$\hat{A}_n = i g_{\rm YM}^{n-2} (2\pi)^4 \delta^{(4)}(P) \left(\langle 13\rangle\langle 32\rangle\langle 24\rangle\langle 41\rangle \, + (2 \leftrightarrow 4)\right) \prod_{i=1}^{n} \frac{1}{\langle \lambda_i, \lambda_{i+1}\rangle} \,, \tag{3.4}$$

where we denoted four scalars by $i = 1, \ldots, 4$.

The MHV amplitudes in component form can be obtained in twistor space as well. Fourier transforming eq. (3.1), the MHV parton amplitude in twistor space is given by

$$\widetilde{A}_n(\lambda_i^a, \mu_i^{\dot\alpha}, \psi_i^A) = i g_{\rm YM}^{n-2} \int {\rm d}^4 x_{\alpha\dot\alpha} {\rm d}^8 \theta_\alpha^A \prod_{i=1}^{n} \delta^{(2)}(\mu_{i\dot\alpha} + x_{\alpha\dot\alpha}\lambda_i^\alpha) \delta^{(4)}(\psi_i^A + \theta_\alpha^A \lambda_i^\alpha) \prod_{j=1}^{n} \frac{1}{\langle \lambda_j, \lambda_{j+1}\rangle} . \tag{3.5}$$

By expanding the fermionic coordinates and picking up suitable terms, one can obtain the MHV amplitudes involving various component fields.

3.2 Multi-parton amplitudes for quiver gauge groups

Consider a quiver-type gauge theory with product gauge group ${\rm SU}(N_1)\times{\rm SU}(N_2)$. Consider also quarks which transform in the bi-fundamental representation $(\mathbf{N}_1, \overline{\mathbf{N}_2})$ under these gauge groups and their complex conjugates. Then, the full multi-parton amplitudes $\mathcal{M}$ involving a quark-antiquark pair with n_1 gauge bosons of ${\rm SU}(N_1)$ and $\widetilde{n}_2$ gauge bosons of ${\rm SU}(N_2)$ can be written as [16]

$$\begin{aligned} \mathcal{M}(q, 1, \ldots, n_1; \widetilde{1}, \ldots, \widetilde{n}_2, \overline{q}) \; = & \sum_{P(n_1), P(n_2)} (X^1 \cdots X^{n_1})_{ij} (Y^1 \cdots Y^{n_2})_{\overline{ji}} \times \\ & \qquad \times \mathcal{A}_{N_1,N_2}(\overline{q}, 1, \ldots, n_1, q; q, \widetilde{1}, \ldots, \widetilde{n}_2, \overline{q}) \,. \end{aligned} \tag{3.6}$$

Here, the sum is over all permutations of n_1 gauge bosons of ${\rm SU}(N_1)$ gauge group between the quark and the antiquark and similarly of $\widetilde{n}_2$ gauge bosons of ${\rm SU}(N_2)$ gauge group between the antiquark and the quark. The $ij, \overline{ji}$ indices refer to ${\rm SU}(N_1)$ and ${\rm SU}(N_2)$ 'color' indices of the quark-antiquark pair. X^A, Y^B are generators of ${\rm SU}(N_1)$ and ${\rm SU}(N_2)$ gauge groups, respectively.

The sub-amplitudes $\mathcal{A}_{N_1,N_2}$ defined as in eq. (3.6) are readily obtained in terms of the multi-parton amplitudes $\mathcal{A}$ involving a quark-antiquark pair:

$$\mathcal{A}_{N_1,N_2}(\overline{q}, 1, \ldots, n_1, q; q, \widetilde{1}, \ldots, \widetilde{n}_2, \overline{q}) = \sum_I \mathcal{A}(q, 1, \ldots, n_1; \widetilde{1}, \ldots, \widetilde{n}_2, \overline{q}) \,, \tag{3.7}$$

where the sum over I refers to over all possibilities the gauge bosons of the second gauge group can be interspersed within those of the first gauge group, while maintaining the order of both the first and the second set of gauge bosons. This sum renders all Feynman diagrams which connect directly the gauge bosons of $\mathrm{SU}(N_1)$ with those of $\mathrm{SU}(N_2)$ to be cancelled.

Concretely, consider the scattering in which the quark has $-$ helicity and the gauge boson α of either gauge group has $-$ helicity while all other partons have $+$ helicities. Then, the corresponding multi-parton amplitude is given by

$$\begin{aligned}\mathcal{A}_{N_1,N_2}(\overline{q},1,\ldots,n_1,q;q,\widetilde{1},\ldots,\widetilde{n}_2,\overline{q}) &= i\frac{\langle q\alpha\rangle^3\langle\overline{q}\alpha\rangle}{\langle q\overline{q}\rangle^2}\sum_I \frac{\langle\overline{q}q\rangle}{\langle\overline{q}1\rangle\langle 12\rangle\cdots\langle n_1\widetilde{1}\rangle\langle\widetilde{1}\widetilde{2}\rangle\cdots\langle\widetilde{n}_2\overline{q}\rangle} \\ &= i\frac{\langle q\alpha\rangle^3\langle\overline{q}\alpha\rangle}{\langle\overline{q}q\rangle^2}\frac{\langle\overline{q}q\rangle}{\langle\overline{q}1\rangle\langle 12\rangle\cdots\langle n_1 q\rangle}\frac{\langle\overline{q}q\rangle}{\langle q\widetilde{1}\rangle\langle\widetilde{1}\widetilde{2}\rangle\cdots\langle\widetilde{n}_2\overline{q}\rangle}, \qquad (3.8)\end{aligned}$$

where in the second expression we used the relation eq. (3.7). In subsection 3.4, we will reproduce this amplitude eq. (3.8) directly from the amplitudes which descends from topological B-model on $\mathbb{CP}^{3|4}$ by taking the super-orbifold projections we identified in previous sections.

3.3 U(1) gauge group

The multi-parton amplitudes involving abelian gauge group are also of interest from various viewpoints. First, quiver gauge theories constructed out of D3-branes contain U(1) subgroups. Among these, overall U(1) subgroup decouples from the rest. The decoupling is obvious from D3-brane viewpoint, but confirmation of such decoupling within the multiparton amplitudes would constitute an interesting checkpoint of the super-orbifold prescription proposed in this work. The rest are relative U(1) groups, and they have non-trivial multiparton amplitudes involving charged particles. Derivation of these amplitudes from topological B-model would offer another interesting checkpoint. Second, from phenomenological viewpoint, the Standard Model contains U(1) hypercharge interactions, and it is of interest to model high-energy scattering mediated by U(1) hypercharge interactions via multiparton amplitudes of the relative U(1) groups. With such motivations, we consider U(1) multiparton amplitude involving a pair of quark-antiquark.

The multi-parton amplitudes for U(1) gauge group is extremely simple, and they are obtainable by setting the gauge group generators to those for the U(1) subgroup of interest. Thus, the resulting MHV amplitude is given by

$$\mathcal{A}_{\mathrm{U}(1)}(q,1,\ldots,n,\overline{q}) = \sum_P \mathcal{A}(q,1,\ldots,n,\overline{q})\,,$$

where the sum runs over *all* permutations of the gauge bosons. It has the remarkable effect of causing all nonabelian Feynman diagrams to cancel one another.

Consider the scattering of n photons with a quark-antiquark pair in which the quark and the photon labeled as α have $-$ helicity and all other partons have $+$ helicity. The amplitude is then described by

$$\mathcal{A}_{\mathrm{U}(1)}(q,1,\ldots,n,\overline{q}) = i\frac{\langle q\alpha\rangle^3\langle\overline{q}\alpha\rangle}{\langle q\overline{q}\rangle^2}\sum_P \frac{\langle\overline{q}q\rangle}{\langle q1\rangle\langle 12\rangle\cdots\langle n\overline{q}\rangle}$$

$$= i\frac{\langle q\alpha\rangle^3\langle\overline{q}\alpha\rangle}{\langle q\overline{q}\rangle^2}\prod_i \frac{\langle\overline{q}q\rangle}{\langle qi\rangle\langle i\overline{q}\rangle}\,. \tag{3.9}$$

Here, in the second expression, we have used the eikonal-like identity:

$$\sum_{P_n}\frac{\langle p\overline{p}\rangle}{\langle p1\rangle\langle 12\rangle\cdots\langle n\overline{p}\rangle} = \prod_{a=1}^{n}\frac{\langle p\overline{p}\rangle}{\langle pa\rangle\langle a\overline{p}\rangle}\,, \tag{3.10}$$

where we emphasize that the permutation P_n involves only n particles labeled by $1, 2, \ldots, n$. This identity follows straightforwardly from repeated use of the Fierz identity: $\langle a\overline{a}\rangle\langle b\overline{b}\rangle = \langle a\overline{b}\rangle\langle b\overline{a}\rangle + \langle ab\rangle\langle\overline{a}\overline{b}\rangle$. The eikonal-like form of the amplitude eq. (3.9) reflects the physics that multiple soft photons are emitted incoherently one another.

It is then straightforward to combine this result with that of the previous subsection. Consider the gauge group $\mathrm{SU}(N_1)\times\mathrm{SU}(N_2)\times\mathrm{U}(1)$, and this can be viewed as $\mathrm{U}(N_1)\times\mathrm{U}(N_2)/\,\mathrm{U}_d(1)$ where the quotient is the overall $\mathrm{U}(1)$ group. Consider MHV amplitudes for a quark-antiquark pair scattering with n_1 $\mathrm{SU}(N_1)$ gauge bosons, n_2 $\mathrm{SU}(N_2)$ gauge bosons, and n $\mathrm{U}(1)$ photons. It is straightforward to express the amplitude as

$$\begin{aligned}&\mathcal{A}_{N_1,N_2,\mathrm{U}(1)}\left(\overline{q}, 1\ldots, n_1, q; q, \widetilde{1}, \ldots, \widetilde{n}_2, \overline{q}; q, \widehat{1}, \ldots, \widehat{n}, \overline{q}\right) = \\ &\qquad = i\frac{\langle q\alpha\rangle^3\langle\overline{q}\alpha\rangle}{\langle q\overline{q}\rangle^2}\frac{\langle\overline{q}q\rangle}{\langle\overline{q}1\rangle\langle 12\rangle\cdots\langle n_1 q\rangle}\frac{\langle\overline{q}q\rangle}{\langle q\widetilde{1}\rangle\langle\widetilde{1}\widetilde{2}\rangle\cdots\langle\widetilde{n}_2\overline{q}\rangle}\prod_{a=1}^{n}\frac{\langle\overline{q}q\rangle}{\langle q\widehat{a}\rangle\langle\widehat{a}\overline{q}\rangle}\,. \end{aligned}\tag{3.11}$$

Here, as above, we assigned the helicities so that the quark q and α-th gauge boson (out of $n_1 + n_2 + n$ gauge bosons involved) carry $-$ helicity while all other particles carry $+$ helicities.

In the following subsection, from the topological string amplitudes on super-orbifold $\mathbb{CP}^{3|4}/\mathbb{Z}_2$, we will reproduce these amplitudes involving $\mathrm{U}(1)$ subgroups as well.

3.4 $\mathrm{U}(N)\times\mathrm{U}(N)$ parton amplitudes from topological B-model orbifold

Having constructed topological strings on super-orbifolds that correspond to quiver gauge theories with product gauge groups and lower supersymmetries, parton amplitudes can be computed straightforwardly from Witten's topological B-model amplitudes, which are the same as $\mathcal{N}=4$ super Yang-Mills multiparton amplitudes, by taking into account of appropriate orbifold projections.

We will now compute some of such amplitudes, in particular, those involving bi-fundamental fermions and show that they reproduce those gauge theory amplitudes summarized in the previous subsections. This is actually a simple matter once we show that such amplitudes can be obtained from $\mathcal{N}=4$ multiparton amplitudes with a suitable orbifold projection. Since the topological strings on $\mathbb{CP}^{(3|4)}$ reproduces the $\mathcal{N}=4$ multiparton amplitudes and since the same orbifold action was used for defining the corresponding topological strings, it follows that the multiparton amplitudes involving bi-fundamental fermions could be obtained from the topological strings on a suitable super-orbifold.

To demonstrate this, we take the $\mathcal{N}=2$ quiver gauge theory considered in the previous section. After the $\mathbb{Z}_2$ orbifold projection, the gauge groups were $\mathrm{U}(N)_1\times\mathrm{U}(N)_2$. We will begin with amplitudes involving nonabelian gauge bosons. Denote in matrix notation the gauge bosons of $\mathrm{SU}(N)_1$ group by

$$\lambda_1\Gamma \qquad \text{where} \qquad \Gamma = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \tag{3.12}$$

and gauge bosons of $\mathrm{SU}(N)_2$ group by

$$\lambda_2\overline{\Gamma} \qquad \text{where} \qquad \overline{\Gamma} = \begin{pmatrix} 0 & 0 \\ 0 & Y \end{pmatrix} \tag{3.13}$$

The quarks transforming as $(N, \overline{N})$ are denoted by

$$qQ \qquad \text{where} \qquad Q = \begin{pmatrix} 0 & W \\ 0 & 0 \end{pmatrix}, \tag{3.14}$$

while antiquarks transforming as $(\overline{N}, N)$ are denoted by

$$\overline{q}\overline{Q} \qquad \text{where} \qquad \overline{Q} = \begin{pmatrix} 0 & 0 \\ W^\dagger & 0 \end{pmatrix}. \tag{3.15}$$

Rather than working directly with the topological B-model amplitudes, since they are the same as multiparton amplitudes in $\mathcal{N} = 4$ supersymmetric gauge theory, we will work with the latter. The multi-parton amplitudes in the quiver gauge theory are then obtainable straightforwardly from the covering $\mathcal{N} = 4$ multi-parton amplitudes in which all of these particles correspond to various components of the parent gauge boson. Thus, it is a simple matter to obtain the MHV amplitude with two external fermions and $m + n$ gluons, where m, n are the number of gauge bosons of $\mathrm{SU}(N)_1$ and $\mathrm{SU}(N)_2$ groups, respectively. We find, stripping off the gauge coupling constant and overall momentum conserving delta functions,

$$\begin{aligned} \mathcal{A}_{\text{quiver}} \equiv i \sum_{P_{m+n+2}} & \mathrm{Tr}\left(\Gamma_1\Gamma_2\cdots\Gamma_m Q\overline{\Gamma}_1\overline{\Gamma}_2\cdots\overline{\Gamma}_m\bar{Q}\right) \times \\ & \times \frac{\langle q\alpha\rangle^3\langle\overline{q}\alpha\rangle}{\langle\overline{q}1\rangle\langle 12\rangle\langle 23\rangle\cdots\langle mq\rangle\langle q\widetilde{1}\rangle\langle\widetilde{1}\widetilde{2}\rangle\cdots\langle\widetilde{n}\overline{q}\rangle}. \end{aligned} \tag{3.16}$$

Here, P_{m+n+2} descends directly from the definition of multiparton amplitudes in the covering $\mathcal{N} = 4$ super Yang-Mills theory, and hence involves permutations of all $m + n + 2$ partons. However, one readily observes that permutations directly connecting gauge bosons of $\mathrm{SU}(N)_1$ and $\mathrm{SU}(N)_2$ vanishes upon taking the color trace. Thus the resulting expression reduces precisely to the same one as the gauge theory result eq. (3.6), where the kinematical part of the amplitude agrees precisely with the second expression in eq. (3.8).

A comment is in order. The MHV amplitudes we derived in eq. (3.16) actually holds for $\mathrm{SU}(N_1)\times\mathrm{SU}(N_2)$ for arbitrary N_1, N_2 with $N_1+N_2 = 2N$. This is because the conformality condition $\mathrm{Tr}\,\gamma_\Gamma = 0$ enters beginning at one-loop order. Thus, agreement of eq. (3.16) with eq. (3.8) is exact for all tree-level MHV amplitudes.

We can also examine multiparton amplitudes for U(1) subgroups. This amounts to replacing the generators for gauge bosons and quark-antiquark pair by those describing a

U(1) subgroup of interest. Thus, for overall U(1) subgroup, which we know to be decoupled and would yield vanishing amplitude, we would replace $\Gamma, \overline{\Gamma}$ in eqs. (3.12), (3.13) by

$$\Gamma, \overline{\Gamma} \to \begin{pmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix}.$$

Then, the color factor is reduced to an overall constant $\mathrm{Tr}\, WW^\dagger$ and picks up the color indices of quark-antiquark pair, so the amplitude is reduced to sum over permutations of kinematical amplitude in eq. (3.16). Utilizing the eikonal-like identity eq. (3.10), we can express

$$\sum_{P_m, P_n} \frac{1}{\langle \overline{q} 1 \rangle \cdots \langle m q \rangle \langle q \widetilde{1} \rangle \cdots \langle \widetilde{n} \overline{q} \rangle} = (-)^n \prod_{a=1}^{m+n} \frac{\langle q \overline{q} \rangle}{\langle q a \rangle \langle a \overline{q} \rangle}. \tag{3.17}$$

Notice the sign factor $(-)^n$ in the final expression. It originates from reversing the order $\langle q1 \rangle \cdots \langle n\overline{q} \rangle$ into $\langle \overline{q} n \rangle \cdots \langle 1 q \rangle$ so that the identity eq. (3.10) can be applied uniformly for all $m + n$ U(1) gauge bosons. Since $\Gamma = \overline{\Gamma}$, we will need to consider other amplitudes proportional to the color factor:

$$\mathrm{Tr} \left(\Gamma_1 \cdots \Gamma_p Q \overline{\Gamma}_1 \cdots \overline{\Gamma}_q \overline{Q} \right) \qquad \text{for } (p + q) = (m + n) \tag{3.18}$$

which all result in the same form of the amplitude. Summing eq. (3.17) over all such possibilities, the sign factor $(-)^n$ conspires with the combinatorial factors such that the total sum vanishes identically. This confirms our anticipation that the overall U(1) subgroup decouples directly at the level of multiparton amplitudes. Notice that the sign factor $(-)^n$, which originates from the kinematical part of the amplitude was crucial for leading to such a nullifying result.

For the relative U(1) gauge subgroup, the generators for gauge bosons are now to be replaced by

$$\Gamma, \overline{\Gamma} \to \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}.$$

Again, the color factor is reduced to an overall constant, but now its value is weighted by an extra sign factor and yields $(-)^n\, \mathrm{Tr}\, WW^\dagger$. More specifically, the extra sign factor $(-)^n$ arises as one passes Q through $\overline{\Gamma}_1 \cdots \overline{\Gamma}_n$ and place it next to $\overline{Q}$. Thus, in repeating the same combinatorial considerations, this sign factor cancels the sign factor $(-)^n$ that originated from the kinematical amplitudes. As such, upon summing over all amplitudes involving $(m + n)$ gauge bosons, we obtain a nonvanishing amplitude, which is the same as gauge theory result eq. (3.9), up to relative normalization of gauge coupling contants.

It is also straightforward to put all these together, and it is readily seen that the most general multiparton amplitudes involving quark-antiquark pair as obtained from topological strings theory via super-orbifolding agrees with the field theory result eq. (3.11).

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